

Hecke-Symmetry and Rational Period Functions

Dedicated to the memory of my teacher Marvin Knopp,
who corrected many of my mistakes.

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Abstract

In this paper we continue work in the direction of a characterization of rational period functions on the Hecke groups. We examine the role that Hecke-symmetry of poles plays in this setting, and pay particular attention to non-symmetric irreducible systems of poles for a rational period function. This gives us a new expression for a class of rational period functions of any positive even integer weight on the Hecke groups. We illustrate these properties with examples of specific rational period functions. We also correct the wording of a theorem from an earlier paper.

1 Introduction

Marvin Knopp first defined and studied rational period functions (RPFs) for automorphic integrals [Kno74] and gave the first example of an RPF with nonzero poles [Kno78].

Knopp [Kno81], Hawkins¹ and Choie and Parson [CP90, CP91] took the main steps toward an explicit characterization of RPFs on the modular group $\Gamma(1)$. Ash [Ash89] gave an abstract characterization, and then Choie and Zagier [CZ93] and Parson [Par93] provided a more explicit characterization of the RPFs on $\Gamma(1)$. The explicit characterization uses continued fractions to establish a connection between the poles of RPFs and binary quadratic forms.

Schmidt [Sch96] generalized Ash's work, giving an abstract characterization of RPFs on any finitely generated Fuchsian group of the first kind with parabolic elements, a class of groups which includes the Hecke groups. Schmidt [Sch93] and Schmidt and Sheingorn [SS95] took steps toward an explicit characterization of RPFs on the Hecke groups using generalizations of classical continued fractions and binary quadratic forms. This author continued that work in [CR01] and [Res09].

¹“On rational period functions for the modular group,” unpublished manuscript.

The nonzero poles of RPFs on $\Gamma(1)$ have unique algebraic conjugates because they are quadratic irrational numbers. The nonzero poles of RPFs on a Hecke group are algebraic numbers, usually of degree greater than 2. In [CR01] and [Res09] we define the unique Hecke-conjugate for any nonzero pole of an RPF on a Hecke group. In this paper we explore the roles that Hecke-conjugation and the related Hecke-symmetry of pole sets play in constructing RPFs on the Hecke groups.

2 Background

In this section we list definitions and results necessary for working with Hecke-symmetry and rational period functions. More details can be found in [CR01] and [Res09].

2.1 Hecke groups and Hecke-symmetry

Throughout this paper we fix an integer $p \geq 3$ and $\lambda = \lambda_p = 2 \cos(\pi/p)$. Put $S = S_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We define *Hecke group*

$$G_p = G(\lambda_p) = \langle S, T \rangle / \{\pm I\}.$$

If we put $U = U_\lambda = S_\lambda T = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}$ we can write the group relations of G_p as $T^2 = U^p = I$. It is well-known that $G_p = \text{PSL}(2, \mathbb{Z}[\lambda_3]) = \Gamma(1)$; however for the other Hecke groups $G_p \subsetneq \text{PSL}(2, \mathbb{Z}[\lambda_p])$. The entries of elements of G_p are in $\mathbb{Z}[\lambda_p]$, the ring of algebraic integers for $\mathbb{Q}(\lambda_p)$.

Members of G_p act on the Riemann sphere as Möbius transformations. An element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_p$ is *hyperbolic* if $|a + d| > 2$, *parabolic* if $|a + d| = 2$, and *elliptic* if $|a + d| < 2$. We designate fixed points accordingly. Hyperbolic Möbius transformations each have two distinct real fixed points.

The *stabilizer* in G_p of a complex number z , $\text{stab}(z) = \{M \in G_p : Mz = z\}$, is a cyclic subgroup of G_p . We define the *Hecke-conjugate* of any hyperbolic fixed point of G_p to be the other fixed point of the elements in its stabilizer. We denote the Hecke-conjugate of α by α' . If R is a set of hyperbolic fixed points of G_p we write $R' = \{x' : x \in R\}$. We say that a set R has *Hecke-symmetry* if $R = R'$. We note that $R \cup R'$ has Hecke-symmetry for any set of hyperbolic points R .

2.2 λ -binary quadratic forms

Hawkins² pointed out a deep connection between rational period functions for the modular group and classical binary quadratic forms. We exploit a similar connection between rational period functions for Hecke groups and generalized binary quadratic forms.

²“On rational period functions for the modular group,” unpublished manuscript.

We let $\mathcal{Q}_{p,D}$ denote the set of binary quadratic forms

$$Q(x, y) = Ax^2 + Bxy + Cy^2,$$

with coefficients in $\mathbb{Z}[\lambda_p]$, and discriminant D . We also denote a form by $Q = [A, B, C]$ and refer to it as a λ -BQF. We restrict our attention to indefinite forms.

We define an action of G_p on $\mathcal{Q}_{p,D}$ by $(Q \circ M)(x, y) = Q(ax + by, cx + dy)$ for $Q \in \mathcal{Q}_{p,D}$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_p$. This action preserves the discriminant and partitions $\mathcal{Q}_{p,D}$ into equivalence classes of forms.

In [Res09] we describe a one-to-one correspondence between hyperbolic fixed points of G_p and certain λ -BQFs. We first use a variant of Rosen's λ -continued fractions [Ros54, Sch93] to map every hyperbolic point α to the unique primitive hyperbolic element $M_\alpha \in G_p$ with positive trace that has α as an attracting fixed point. This mapping associates Hecke-conjugates with inverse elements in the Hecke group, that is, $M_{\alpha'} = M_\alpha^{-1}$. We also map every hyperbolic element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_p$ with positive trace to a unique indefinite λ -BQF Q such that the roots of $Q(z, 1) = cz^2 + (d-a)z - b$ are the fixed points of M . Every domain element for this map is hyperbolic, so we call the images *hyperbolic* λ -BQFs.

The composition of the two maps described above associates every hyperbolic fixed point α with a unique hyperbolic λ -BQF Q_α . These mappings are both injective, and the inverse of the composition associates every hyperbolic quadratic form $Q = [A, B, C]$ with the hyperbolic number $\alpha_Q = \frac{-B + \sqrt{D}}{2A}$, where D is the discriminant of Q .

Every equivalence class of λ -BQFs contains either all hyperbolic forms or no hyperbolic forms, so we may label equivalence classes themselves as hyperbolic or non-hyperbolic.

If \mathcal{A} denotes an equivalence class of λ -BQFs, we put $-\mathcal{A} = \{-Q \mid Q \in \mathcal{A}\}$. Then $-\mathcal{A}$ is another equivalence class of forms, not necessarily distinct from \mathcal{A} . If \mathcal{A} is hyperbolic, so is $-\mathcal{A}$, and the numbers associated with the forms in $-\mathcal{A}$ are the Hecke-conjugates of the numbers associated with the forms in \mathcal{A} .

We call a hyperbolic λ -BQF $Q = [A, B, C]$ *G_p -simple* if $A > 0 > C$. If Q is a simple λ -BQF, we say that the associated hyperbolic number α_Q is a *G_p -simple* number. A hyperbolic number α is simple if and only if $\alpha' < 0 < \alpha$.

If \mathcal{A} is a hyperbolic equivalence class of λ -BQFs we write $Z_{\mathcal{A}} = \{x : Q_x \in \mathcal{A}, Q_x \text{ simple}\}$. These sets are nonempty; every hyperbolic equivalence class of λ -BQFs contains at least one simple form.

2.3 Rational period functions

For $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_p$ and $f(z)$ a function of a complex variable, we define the *weight $2k$ slash operator* $f|_{2k} M = f| M$ to be

$$(f| M)(z) = (cz + d)^{-2k} f(Mz).$$

Definition 1. Fix $p \geq 3$ and let $k \in \mathbb{Z}^+$. A *rational period function (RPF)* of weight $2k$ for G_p is a rational function that satisfies the relations

$$q + q \mid T = 0, \quad (1)$$

and

$$q + q \mid U + \cdots + q \mid U^{p-1} = 0. \quad (2)$$

This definition is equivalent to Marvin Knopp's original definition of rational period functions for automorphic integrals [Kno74].

For any rational period function of weight $2k$ for G_p we let $P(q)$ denote the set of poles of q . Hawkins defined an *irreducible system of poles (ISP)*, which is the minimal set of poles forced to occur together by the relations (1) and (2).

The poles of an RPF on G_p are all real, and the nonzero poles are all hyperbolic fixed points of G_p . The set of positive poles in an ISP is $\mathcal{Z}_{\mathcal{A}}$ for some hyperbolic equivalence class \mathcal{A} . The ISP associated with \mathcal{A} is

$$\begin{aligned} P_{\mathcal{A}} &= \mathcal{Z}_{\mathcal{A}} \cup T\mathcal{Z}_{\mathcal{A}} \\ &= \mathcal{Z}_{\mathcal{A}} \cup \mathcal{Z}'_{-\mathcal{A}}. \end{aligned}$$

If q is an RPF of weight $2k$ on G_p with a pole *only* at zero, then q must have the form [MR84]

$$q_{k,0}(z) = \begin{cases} a_0(1 - z^{-2k}), & \text{if } 2k \neq 2, \\ a_0(1 - z^{-2}) + b_1 z^{-1}, & \text{if } 2k = 2. \end{cases} \quad (3)$$

If $\alpha \neq 0$ occurs as a pole of an RPF q of weight $2k \in 2\mathbb{Z}^+$ on G_p , we define

$$q_{k,\alpha}(z) = PP_{\alpha} \left[\frac{D^{k/2}}{Q_{\alpha}(z, 1)^k} \right] = PP_{\alpha} \left[\frac{(\alpha - \alpha')^k}{(z - \alpha)^k (z - \alpha')^k} \right], \quad (4)$$

where D is the discriminant of the λ -BQF Q_{α} . With this we have the following expression, valid for any RPF on G_p [CR01, p. 292].

Theorem 1. *An RPF of weight $2k \in 2\mathbb{Z}^+$ on G_p is of the form*

$$q(z) = \sum_{\ell=1}^L C_{\ell} \left(\sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_{\ell}}} q_{k,\alpha}(z) - \sum_{\alpha \in \mathcal{Z}_{-\mathcal{A}_{\ell}}} q_{k,\alpha'}(z) \right) + c_0 q_{k,0}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n}, \quad (5)$$

where each \mathcal{A}_{ℓ} is a G_p -equivalence class of λ -BQFs, $\mathcal{Z}_{\mathcal{A}_{\ell}}$ is the cycle of positive poles associated with \mathcal{A}_{ℓ} , $q_{k,\alpha}$ is given by (4), $q_{k,0}$ is given by (3), and the C_{ℓ} and c_n are all constants.

3 RPFs with Hecke-symmetric pole sets

An RPF must have a particularly simple form if every irreducible system of poles has Hecke-symmetry, and an RPF may have a simple form if its complete set of poles has Hecke-symmetry.

3.1 A correction

In Theorem 2 of [CR01] we consider the case of pole sets that have Hecke-symmetry, with k odd. The converse in Theorem 2 of is incorrect as stated. The mis-statement involves the distinction between

- (A) an RPF for which every nonzero irreducible system of poles has Hecke-symmetry, and
- (B) an RPF for which the complete set of nonzero poles has Hecke-symmetry.

An RPF that satisfies (A) must satisfy (B), but the converse does not hold. We state a corrected theorem that accounts for this distinction.

Theorem 2 (corrected). *Suppose that k is an odd positive integer.*

Suppose q is an RPF of weight $2k$ on G_p with Hecke-symmetric irreducible systems of poles. Then q is of the form

$$q(z) = \sum_{\ell=1}^L d_\ell \sum_{\alpha \in Z_{\mathcal{A}_\ell}} Q_\alpha(z, 1)^{-k} + c_0 q_{k,0}(z), \quad (6)$$

where each \mathcal{A}_ℓ is a G_p -equivalence class of λ -BQFs satisfying $-\mathcal{A}_\ell = \mathcal{A}_\ell$, the d_ℓ ($1 \leq \ell \leq M$) are constants, and $q_{k,0}(z)$ is given by (3).

Conversely, any rational function of the form (6) is an RPF of weight $2k$ on G_p with a set of nonzero poles that (taken in its entirety) is Hecke-symmetric.

Proof. The proof in [CR01] is correct for this statement of the theorem. \square

The corrected converse statement leaves open the possibility that there exist rational period functions with Hecke-symmetric pole sets that do not have Hecke-symmetric irreducible systems of poles. Indeed, such rational period functions exist, as the next example shows.

Example 1. Put $\lambda = \lambda_4 = \sqrt{2}$. The smallest λ_4 -BQF discriminant with class number greater than 1 is $D = 14$, which has class number $h_{4,14} = 2$ [HR13]. The two equivalence classes satisfy $\mathcal{A} \neq -\mathcal{A}$, so we have $\mathcal{A}_1 = -\mathcal{A}_2$. We denote by \mathcal{A}_1 the equivalence class containing the simple BQFs

$$Q_{1,1} = [1, -\sqrt{2}, -3] \text{ and } Q_{1,2} = [1, \sqrt{2}, -3],$$

and by \mathcal{A}_2 the equivalence class containing the simple BQFs

$$Q_{2,1} = [3, -\sqrt{2}, -1] \text{ and } Q_{2,2} = [3, \sqrt{2}, -1].$$

The corresponding positive poles are

$$\mathcal{Z}_{\mathcal{A}_1} = \{\alpha_1, \alpha_2\} = \left\{ \frac{\sqrt{2} + \sqrt{14}}{2}, \frac{-\sqrt{2} + \sqrt{14}}{2} \right\},$$

and

$$\mathcal{Z}_{\mathcal{A}_2} = \{\beta_1, \beta_2\} = \left\{ \frac{\sqrt{2} + \sqrt{14}}{6}, \frac{-\sqrt{2} + \sqrt{14}}{6} \right\}.$$

The corresponding negative poles are

$$T\mathcal{Z}_{\mathcal{A}_1} = \left\{ \frac{\sqrt{2} - \sqrt{14}}{6}, \frac{-\sqrt{2} - \sqrt{14}}{6} \right\} = \{\beta'_1, \beta'_2\},$$

and

$$T\mathcal{Z}_{\mathcal{A}_2} = \left\{ \frac{\sqrt{2} - \sqrt{14}}{2}, \frac{-\sqrt{2} - \sqrt{14}}{2} \right\} = \{\alpha'_1, \alpha'_2\}.$$

If we put $L = 2$, $d_1 = d_2 = 1$, and $c_0 = 0$ in (6) we have the rational period function

$$\begin{aligned} q(z) &= Q_{\alpha_1}(z, 1)^{-k} + Q_{\alpha_2}(z, 1)^{-k} + Q_{\beta_1}(z, 1)^{-k} + Q_{\beta_2}(z, 1)^{-k} \\ &= \frac{1}{(z^2 - \sqrt{2}z - 3)^k} + \frac{1}{(z^2 + \sqrt{2}z - 3)^k} + \frac{1}{(3z^2 - \sqrt{2}z - 1)^k} + \frac{1}{(3z^2 + \sqrt{2}z - 1)^k} \\ &= \frac{1}{(z - \alpha_1)^k(z - \alpha'_1)^k} + \frac{1}{(z - \alpha_2)^k(z - \alpha'_2)^k} + \frac{1}{3^k(z - \beta_1)^k(z - \beta'_1)^k} + \frac{1}{3^k(z - \beta_2)^k(z - \beta'_2)^k}. \end{aligned}$$

The set of poles

$$P(q) = \{\alpha_1, \alpha'_1, \alpha_2, \alpha'_2, \beta_1, \beta'_1, \beta_2, \beta'_2\},$$

is Hecke-symmetric, while both irreducible systems of poles

$$P_{\mathcal{A}_1} = \mathcal{Z}_{\mathcal{A}_1} \cup T\mathcal{Z}_{\mathcal{A}_1} = \{\alpha_1, \alpha_2, \beta'_1, \beta'_2\},$$

and

$$P_{\mathcal{A}_2} = \mathcal{Z}_{\mathcal{A}_2} \cup T\mathcal{Z}_{\mathcal{A}_2} = \{\beta_1, \beta_2, \alpha'_1, \alpha'_2\},$$

are not Hecke-symmetric.

3.2 Pairs of non-symmetric ISPs

An important feature of the rational period function in Example 1 is that the union of the two non-symmetric irreducible systems of poles is Hecke-symmetric. The next lemma shows that this always happens for RPFs with Hecke-symmetric pole sets, that is, the non-symmetric ISPs always occur in pairs.

Lemma 1. *Let q be an RPF of weight $2k$ on G_p for $k \in \mathbb{Z}^+$ and $p \geq 3$. Suppose that q has a Hecke-symmetric set of nonzero poles. If q has any ISPs that are not Hecke-symmetric, then such ISPs must occur in pairs. That is, if $P_{\mathcal{A}}$ is a non-Hecke-symmetric ISP for q then $P_{-\mathcal{A}}$ is also a non-Hecke-symmetric ISP for q that is distinct from $P_{\mathcal{A}}$, and $P_{\mathcal{A}} \cup P_{-\mathcal{A}}$ is Hecke-symmetric.*

Proof. If $P_{\mathcal{A}}$ is a non-Hecke-symmetric ISP then $\mathcal{A} \neq -\mathcal{A}$ and $P_{\mathcal{A}} \neq P_{-\mathcal{A}}$. Moreover,

$$\begin{aligned} P'_{\mathcal{A}} &= (\mathcal{Z}_{\mathcal{A}} \cup \mathcal{Z}'_{-\mathcal{A}})' \\ &= \mathcal{Z}_{-\mathcal{A}} \cup \mathcal{Z}'_{\mathcal{A}} \\ &= P_{-\mathcal{A}}, \end{aligned}$$

so $P_{-\mathcal{A}} \neq P'_{-\mathcal{A}}$, and $P_{-\mathcal{A}}$ is also non-Hecke-symmetric. On the other hand,

$$P_{\mathcal{A}} \cup P_{-\mathcal{A}} = (\mathcal{Z}_{\mathcal{A}} \cup \mathcal{Z}'_{-\mathcal{A}}) \cup (\mathcal{Z}_{-\mathcal{A}} \cup \mathcal{Z}'_{\mathcal{A}}),$$

which is Hecke-symmetric. \square

3.3 RPFs of any weight

The class of functions given by (6) contains rational period functions for odd values of k (half of the weight). In the next theorem we use sums of powers of λ -BQFs to write a class of functions that contains RPFs for any even integer weight. We first state and prove two lemmas that will help us to specify the poles in the two RPF relations.

Lemma 2. *Let α be a hyperbolic fixed point of G_p for $p \geq 3$, and put $Q_{\alpha}(z) = Q_{\alpha}(z, 1)$. Then for every $M \in G_p$ we have*

$$(Q_{\alpha}^{-k} \mid M)(z) = Q_{M^{-1}\alpha}^{-k}(z).$$

Remark. Schmidt first stated this in [Sch93, p. 236] without proof. We include a proof for completeness.

Proof. Write $Q_{\alpha}(z) = Az^2 + Bz + C = A(z - \alpha)(z - \alpha')$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $Q_{M^{-1}\alpha}(z) = \tilde{A}(z - M^{-1}\alpha)(z - M^{-1}\alpha')$, where $\tilde{A} = Aa^2 + Bac + Cc^2$. We calculate that

$$\begin{aligned} (Q_{\alpha}^{-k} \mid M)(z) &= (cz + d)^{-2k} A^{-k} \left(\frac{az + b}{cz + d} - \alpha \right)^{-k} \left(\frac{az + b}{cz + d} - \alpha' \right)^{-k} \\ &= A^{-k} ((-c\alpha + a)z - (d\alpha - b))^{-k} ((-c\alpha' + a)z - (d\alpha' - b))^{-k} \\ &= A^{-k} (-c\alpha + a)^{-k} (-c\alpha' + a)^{-k} \left(z - \frac{d\alpha - b}{-c\alpha + a} \right)^{-k} \left(z - \frac{d\alpha' - b}{-c\alpha' + a} \right)^{-k} \\ &= A^{-k} (a^2 - (\alpha + \alpha')ac + \alpha\alpha'c^2)^{-k} (z - M^{-1}\alpha)^{-k} (z - M^{-1}\alpha')^{-k} \\ &= (Aa^2 + Bac + Cc^2)^{-k} (z - M^{-1}\alpha)^{-k} (z - M^{-1}\alpha')^{-k} \\ &= Q_{M^{-1}\alpha}^{-k}(z). \end{aligned}$$

\square

The next lemma specifies the poles that result when we apply the second relation to part of the RPF in (5). We consider poles in the extended complex plane, so we include the point at ∞ in our analysis.

Lemma 3. *Let $U = U_{\lambda_p}$ for $p \geq 3$ and let $k \in \mathbb{Z}^+$. Suppose that $r(z) = \sum_{n=1}^{2k-1} \frac{c_n}{z^n}$ is a nontrivial function. Then for $1 \leq t \leq p-2$, the function $(r \mid U^t)(z)$ has poles at $z = U^{p-t}(0)$ and $z = U^{p-t+1}(0)$, and no other poles in $\mathbb{C} \cup \{\infty\}$. The function $(r \mid U^{p-1})(z)$ has a pole at $z = U^2(0)$, and no other poles in $\mathbb{C} \cup \{\infty\}$.*

Proof. If we let $\gamma_t = \frac{\sin(t\pi/p)}{\sin(\pi/p)}$ then $U^t = \begin{pmatrix} \gamma_{t+1} & -\gamma_t \\ \gamma_t & -\gamma_{t-1} \end{pmatrix}$ for $t \in \mathbb{Z}$ [MR84]. For $1 \leq t \leq p-1$ we have

$$(r \mid U^t)(z) = (\gamma_t z - \gamma_{t-1})^{-2k} q(U^t z).$$

The automorphy factor $(\gamma_t z - \gamma_{t-1})^{-2k}$ has a pole of order $2k$ at

$$z = \frac{\gamma_{t-1}}{\gamma_t} = \frac{1}{U^t(0)} = U^{p-t+1}(0).$$

We have used Lemma 9 in [Res09] for the last equality. The automorphy factor has a zero at ∞ of order $2k$. The function $r(U^t z)$ has a pole at $z = U^{p-t}(0)$ of order less than $2k$, and a zero at $z = U^{p-t+1}(0)$ of order less than $2k$, and no other poles or zeros. When we multiply expressions, the zero at $z = U^{p-t+1}(0)$ (from $r(U^t z)$) combines with the pole at $z = U^{p-t+1}(0)$ (from $(\gamma_t z - \gamma_{t-1})^{-2k}$) but does not cancel it because the order of the zero is less than $2k$.

If $t = p-1$, we have the additional combination of the pole at $z = U(0) = \infty$ (from $r(U^{p-1} z)$) with the zero at $z = \infty$ (from $(\gamma_{p-1} z - \gamma_{p-2})^{-2k}$) to give no pole at $z = \infty$. The poles that remain are the ones given in the statement of the lemma. \square

Theorem 3. *Let*

$$q(z) = \sum_{\alpha \in Z_{\mathcal{A}}} Q_{\alpha}(z, 1)^{-k} - (-1)^k \sum_{\alpha \in Z_{-\mathcal{A}}} Q_{\alpha}(z, 1)^{-k}, \quad (7)$$

where each \mathcal{A}_{ℓ} is a G_p -equivalence class of $\mathbb{Z}[\lambda_p]$ -BQFs. Then q is a rational period function of weight $2k$ on G_p for any positive integer k .

Remark.

- (i) This generalizes the class of RPFs on the modular group given in Theorem 3.2 of [CP90], which (with an appropriate change of notation) is also given in Theorem 3 of [DIT10].
- (ii) We can say more about the RPF in (7) for specific cases.

- If $\mathcal{A} = -\mathcal{A}$ and k is odd then

$$q(z) = 2 \sum_{\alpha \in Z_{\mathcal{A}}} Q_{\alpha}(z, 1)^{-k},$$

which is contained in the expression in Theorem 2. This RPF has only one ISP and it is Hecke-symmetric.

- If $\mathcal{A} = -\mathcal{A}$ and k is even then $q(z) \equiv 0$.
- If $\mathcal{A} \neq -\mathcal{A}$ (for any k) then q has two non-symmetric ISPs $P_{\mathcal{A}}$ and $P_{-\mathcal{A}}$, but the complete set of poles $P_{\mathcal{A}} \cup P_{-\mathcal{A}}$ is Hecke-symmetric.

Proof. We choose $L = 2$ in (5), put $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{A}_2 = -\mathcal{A}$, and note that \mathcal{A}_1 and \mathcal{A}_2 have the same discriminant D . If we put $C_1 = D^{-k/2}$, $C_2 = (-1)^{k+1} D^{-k/2}$, and $c_0 = 0$ the RPF given by (5) is

$$\begin{aligned} q(z) &= D^{-k/2} \left(\sum_{\alpha \in Z_{\mathcal{A}}} q_{k,\alpha}(z) - \sum_{\alpha \in Z_{-\mathcal{A}}} q_{k,\alpha'}(z) \right) \\ &\quad + (-1)^{k+1} D^{-k/2} \left(\sum_{\alpha \in Z_{-\mathcal{A}}} q_{k,\alpha}(z) - \sum_{\alpha \in Z_{\mathcal{A}}} q_{k,\alpha'}(z) \right) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n} \\ &= D^{-k/2} \sum_{\alpha \in Z_{\mathcal{A}}} (q_{k,\alpha}(z) + (-1)^k q_{k,\alpha'}(z)) \\ &\quad - D^{-k/2} \sum_{\alpha \in Z_{-\mathcal{A}}} ((-1)^k q_{k,\alpha}(z) + q_{k,\alpha'}(z)) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n}. \end{aligned} \quad (8)$$

The functions $q_{k,\alpha}(z)$ and $q_{k,\alpha'}(z)$ are given by (4), and for $q_{k,\alpha'}(z)$ we calculate that

$$q_{k,\alpha'}(z) = PP_{\alpha'} \left[\frac{(\alpha' - \alpha)^k}{(z - \alpha')^k (z - \alpha)^k} \right] = (-1)^k PP_{\alpha'} \left[\frac{(\alpha - \alpha')^k}{(z - \alpha)^k (z - \alpha')^k} \right],$$

so

$$q_{k,\alpha}(z) + (-1)^k q_{k,\alpha'}(z) = \frac{D^{k/2}}{Q_{\alpha}(z, 1)^k}.$$

With this (8) becomes

$$q(z) = \sum_{\alpha \in Z_{\mathcal{A}}} Q_{\alpha}(z, 1)^{-k} - (-1)^k \sum_{\alpha \in Z_{-\mathcal{A}}} Q_{\alpha}(z, 1)^{-k} + \sum_{n=1}^{2k-1} \frac{c_n}{z^n}.$$

It remains to show that $\sum_{n=1}^{2k-1} \frac{c_n}{z^n} = 0$. To that end we put

$$q_1(z) = \sum_{\alpha \in Z_{\mathcal{A}}} Q_{\alpha}(z, 1)^{-k} - (-1)^k \sum_{\alpha \in Z_{-\mathcal{A}}} Q_{\alpha}(z, 1)^{-k},$$

and

$$q_2(z) = \sum_{n=1}^{2k-1} \frac{c_n}{z^n},$$

so that $q(z) = q_1(z) + q_2(z)$. We suppose by way of contradiction that q_2 is nontrivial. We will show that this implies that q_2 is itself an RPF, which contradicts the fact that q_2 must have the form (3).

Now q_1 and $q_1 \mid T$ do not have poles at $z = 0$, by Lemma 2. On the other hand, the only pole in $\mathbb{C} \cup \{\infty\}$ for q_2 and $q_2 \mid T$ is at $z = 0$. We use these observations along with the fact that q satisfies the first relation (1) to calculate

$$\begin{aligned} 0 &= PP_0 [q + q \mid T] \\ &= PP_0 [q_2 + q_2 \mid T] \\ &= q_2 + q_2 \mid T, \end{aligned}$$

so q_2 also satisfies the first relation (1).

Suppose $1 \leq t \leq p-1$. Now q_1 and $q_1 \mid U^t$ (by Lemma 2) have poles only at hyperbolic points. On the other hand, the only poles in $\mathbb{C} \cup \{\infty\}$ for q_2 and $q_2 \mid U^t$ (by Lemma 3) are at one or two of the parabolic points $\{U^s(0) : 2 \leq s \leq p\}$. We use these observations along with the fact that q satisfies the second relation (2) to calculate

$$\begin{aligned} 0 &= \sum_{2 \leq s \leq p} PP_{U^s(0)} [q + q \mid U + \cdots q \mid U^{p-1}] \\ &= \sum_{2 \leq s \leq p} PP_{U^s(0)} [q_2 + q_2 \mid U + \cdots q_2 \mid U^{p-1}] \\ &= q_2 + q_2 \mid U + \cdots q_2 \mid U^{p-1}, \end{aligned}$$

so q_2 also satisfies the second relation (2). But then q_2 must be an RPF, which is a contradiction. Thus $q_2(z) \equiv 0$, and q has the form given in the statement of the theorem. \square

4 An RPF with a non-symmetric pole set

If \mathcal{A} is an equivalence class of λ -BQFs with $\mathcal{A} \neq -\mathcal{A}$, we can use (5) to write a rational period function that has a single nonzero irreducible system of poles $P_{\mathcal{A}} = \mathcal{Z}_{\mathcal{A}} \cup \mathcal{Z}_{-\mathcal{A}'}$, which does not have Hecke-symmetry. We illustrate this with an example.

Example 2. Put $\lambda = \lambda_4 = \sqrt{2}$ and consider the λ_4 -BQFs of discriminant $D = 14$. We let \mathcal{A} be the equivalence class \mathcal{A}_1 from Example 1, so that (using

the notation from Example 1)

$$\begin{aligned}
P_A &= \mathcal{Z}_A \cup \mathcal{Z}_{-A'} \\
&= \{\alpha_1, \alpha_2, \beta'_1, \beta'_2\}, \\
&= \left\{ \frac{\sqrt{2} + \sqrt{14}}{2}, \frac{-\sqrt{2} + \sqrt{14}}{2}, \frac{\sqrt{2} - \sqrt{14}}{6}, \frac{-\sqrt{2} - \sqrt{14}}{6} \right\}.
\end{aligned}$$

We put $L = 1$, $C_1 = 0$ and $c_0 = 0$ in (5) and have the RPF on G_4

$$\begin{aligned}
q(z) &= \sum_{\alpha \in \mathcal{Z}_A} q_{k,\alpha}(z) - \sum_{\alpha \in \mathcal{Z}_{-A}} q_{k,\alpha'}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n} \\
&= q_{k,\alpha_1}(z) + q_{k,\alpha_2}(z) - q_{k,\beta'_1}(z) - q_{k,\beta'_2}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n}.
\end{aligned}$$

We let $k = 1$ and calculate that

$$q(z) = \frac{1}{z - \alpha_1} + \frac{1}{z - \alpha_2} - \frac{1}{z - \beta'_1} - \frac{1}{z - \beta'_2} + \frac{c_1}{z}.$$

We have from (3) that $\frac{c_1}{z}$ is itself an RPF of weight 2 on G_4 so we may let $c_1 = 0$, and

$$\begin{aligned}
q(z) &= \frac{1}{z - \alpha_1} + \frac{1}{z - \alpha_2} - \frac{1}{z - \beta'_1} - \frac{1}{z - \beta'_2} \\
&= \left(z - \frac{\sqrt{2} + \sqrt{14}}{2} \right)^{-1} + \left(z - \frac{-\sqrt{2} + \sqrt{14}}{2} \right)^{-1} \\
&\quad - \left(z - \frac{\sqrt{2} - \sqrt{14}}{6} \right)^{-1} + \left(z - \frac{-\sqrt{2} - \sqrt{14}}{6} \right)^{-1}
\end{aligned}$$

is an RPF of weight 2 on G_4 with one ISP which does not have Hecke-symmetry.

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